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All real forms of $U_q(sl(4;\mathbb{C}))$ and $D = 4$ conformal quantum algebras*

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Abstract. The star operations and reality conditions for the complex quantum algebra $U_q(sl(4;\mathbb{C}))$ providing real quantum algebras $U_q(o(6-k,k))\, k = 0, 1, 2, 3$ and $U_q(su(3,1))$ are classified. Standard and non-standard star operations are considered. It appears that only four choices of real forms (one with $|q| = 1$, three with q real) provide real Hopf algebra $U_q(su(2,2)) \simeq U_q(o(4,2))$ describing $D = 4$ conformal quantum algebras. We show that only the antipode-extended Cartan–Weyl basis of $U_q(sl(4;\mathbb{C}))$ permits to define real q-deformed $D = 4$ conformal algebra generators. In order to obtain the real $D = 4$ Weyl algebra as Hopf subalgebra of $U_q(su(2,2))$ only the non-standard real forms can be employed.

1. Introduction

Recently the Cartan–Weyl basis of quantum Lie algebra $U_q(sl(4;\mathbb{C}))$ was considered as the framework for the description of q-deformed $D = 4$ conformal algebra [1–5]. Because the algebra $sl(4,\mathbb{C})$ describes complexified $D = 4$ conformal algebra it is important to select real forms of $U_q(sl(4;\mathbb{C}))$ describing q-deformed $D = 4$ conformal algebra $U_q(su(2,2)) \simeq U_q(o(4,2))$. In the paper [2] two real forms of $U_q(o,(4,2))$ were obtained by using non-standard $\oplus$-involution which is an automorphism in both algebra and coalgebra sectors||:

$$(a \cdot b)^{\oplus} = b^{\oplus} \cdot a^{\oplus} \qquad (\Delta(a))^{\oplus} = \Delta'(a^{\oplus}) \tag{1.1}$$

where $\Delta' \equiv \tau \circ \Delta$ (τ is a flip operator). Unfortunately at least for generic q the non-standard $\oplus$ involutions at least for generic q lead to difficulties in the representation theory of $\oplus$ real Hopf algebras (e.g. it is not known how to avoid the indefinite metric in the tensor product representations). In this paper we would like to consider the standard real forms of $U_q(sl(4;\mathbb{C}))$ with the star operation described by + involution satisfying the relations:

$$(a \cdot b)^{+} = b^{+} \cdot a^{+} \qquad (\Delta(a))^{+} = \Delta(a^{+}). \tag{1.2}$$

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|| Four possible types of involutive mappings of the Hopf algebras denoted by +, $\oplus$, $*$ and $\circledast$ were introduced in [5] and subsequently discussed in [6, 7]. The $\oplus$-involution is denoted in [7] as (1,1) and +-involution as (1,0) semilinear mappings.

The difference between the involutions (1.1) and (1.2) appears also in different forms of the compatibility conditions of the star operation with the anitpode S [5–7]:

$$S \circ \oplus = \oplus \circ S \qquad S \circ + \circ S \circ + = 1. \tag{1.3}$$

Curiously enough it was easier to define the real q-deformed conformal generators if we used the non-standard involution (1.1). Using the following $\oplus$ involution of the Cartan–Weyl basis of $U_q(sl(4;\mathbb{C}))$ (for real q)

$$\begin{aligned} h_1^{\oplus} &= -h_3 \qquad & h_2^{\oplus} &= -h_2 \\ e_{\pm 1}^{\oplus} &= e_{\pm 3} \qquad & e_{\pm 2}^{\oplus} &= e_{\pm 2} \end{aligned} \tag{1.4}$$

we were able to show in [2] that the following sequence of inclusions of real Hopf algebras is valid:

$$U_q(o(3,1)) \subset U_q(\mathcal{P}_4 \supset\!\!\!\!+ D) \subset U_q(o(4,2)) \tag{1.5}$$

where $U_q(\mathcal{P}_4 \supset\!\!\!\!+ D)$ denotes real $D = 4$ q-deformed Weyl algebra†. The sequence (1.5) is important from the point of view of possible physical applications.

One can raise an important question whether the sequence (1.5) can be written with reality conditions described by the standard involution (1.2). Unfortunately the answer is negative. We shall show below that there are only four standard + involutions defining q-deformed real $D = 4$ conformal algebra $U_q(o(4,2))$ both having the following properties:

(i) The involutions take out from the Cartan–Weyl basis, i.e. in order to define for $q \neq 1$ the $D = 4$ conformal generators, one has to introduce 21 generators of antipode-extended Cartan–Weyl generators (15 generators of Cartan–Weyl basis and six generators $S(e_{\pm a})$ where $a = 4, 5, 6$).

(ii) The relation (1.5) is not valid, i.e. the real q-deformed conformal algebras defined by means of standard involutions can not be used for defining the q-deformed Poincare and Weyl algebras.

The plan of our paper is the following. Firstly in section 2 we present the Cartan–Weyl basis of complex quantum algebra $U_q(sl(4;\mathbb{C}))$, i.e. provide the algebra, coproducts and antipodes for 15 generators h_j, $e_{\pm A}(A = 1 \ldots 6)$. The action of antipode S introduces 21 generators of antipode-extended Cartan–Weyl basis‡. In section 3 following Twietmeyer [8] we provide the description of real Hopf algebras $U_q(o(6-k,k))$ $(k = 0,1,2,3)$ and $U_q(su(3,1))$ as real forms of $U_q(sl(4;\mathbb{C}))$. It appears that many reality conditions take out from the Cartan–Weyl basis, and we would like to stress that all the real forms defining $U_q(o(4,2))$ are of this type. In section 4 we introduce three involutive automorphisms of $U_q(sl(4;\mathbb{C}))$:

(i) Q-automorphisms (type $*$) describing mapping $q \to q^{-1}$;

(ii) Ω-automorphism (type $\circledast$) exchanging the first and third root in the Dynkin diagram for $sl(4;\mathbb{C})$ (q unchanged);

(iii) T-automorphism (type $\oplus$) reversing the order of operators in products (q unchanged).

We shall show that having these three operations one can obtain all the other involutions of $U_q(sl(4;\mathbb{C}))$ belonging to the four types of involutive automorphisms discussed in [5,7]. In section 5 we shall discuss the reality conditions for the universal R-matrix of $D = 4$ q-deformed conformal algebra. In section 6 we shall present final comments.

† We denote by $\mathcal{P}_4$ the $d = 4$ Poincaré algebra and by D the dilatation operator.

‡ In section 2 we shall present the formulae from [2] with the error in the coproduct sector corrected.

2. Antipode-extended Cartan–Weyl basis for $U_q(sl(4;\mathbb{C}))$

The Cartan–Cheveley basis of $U_q(sl(4;\mathbb{C}))$ is given by the formulae:

$$
\begin{aligned}
&[h_j, h_k] = 0 \\
&[h_j, e_{\pm k}] = \pm a_{jk} e_{\pm k} \\
&[e_j, e_{-k}] = \delta_{jk} \frac{q^{h_j} - q^{-h_j}}{q - q^{-1}} \equiv \delta_{jk}[h_j]_q
\end{aligned}
\qquad a_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{2.1}
$$

where h_j describe the Cartan subalgebra and e_j, e_{-j} ($j = 1, 2, 3$) are the generators corresponding to simple roots. Let us define the generators corresponding to non-simple roots as follows [9, 10]:

$$
\begin{aligned}
&e_4 \equiv [e_1, e_2]_q \qquad && e_{-4} \equiv [e_{-2}, e_{-1}]_{q^{-1}} \\
&e_5 \equiv [e_2, e_3]_q \qquad && e_{-5} \equiv [e_{-3}, e_{-2}]_{q^{-1}} \\
&e_6 \equiv [e_1, e_5]_q \qquad && e_{-6} \equiv [e_{-5}, e_{-1}]_{q^{-1}}
\end{aligned} \tag{2.2}
$$

where $[A, B]_x \equiv AB - xBA$. It appears that relations (2.1) can be extended as follows ($A, B = 1 \ldots 6$)

$$
[h_A, e_{\pm A}] = \pm a_{AB} e_{\pm B} \qquad a_{AB} = \begin{pmatrix} 2 & -1 & 0 & 1 & -1 & 1 \\ -1 & 2 & -1 & 1 & 1 & 0 \\ 0 & -1 & 2 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \tag{2.3}
$$

where $h_4 \equiv h_1 + h_2$, $h_5 \equiv h_2 + h_3$, $h_6 \equiv h_1 + h_5$ and

$$
[e_a, e_{-a}] = \frac{q^{h_a} - q^{-h_a}}{q - q^{-1}} \equiv [h_a] \qquad a = 4, 5, 6. \tag{2.4}
$$

We can write down also all the remaining relations between generators of $U_q(sl(4;\mathbb{C}))$. For generators corresponding to positive roots we obtain:

$$
\begin{aligned}
&[e_2, e_4]_q = 0 \qquad && [e_1, e_6]_{q^{-1}} = 0 \\
&[e_1, e_3] = 0 && [e_2, e_5]_{q^{-1}} = 0 \\
&[e_4, e_3]_q = e_6 && [e_3, e_5]_q = 0 \\
&[e_1, e_4]_{q^{-1}} = 0 && [e_2, e_6] = 0 \\
& && [e_3, e_6]_q = 0
\end{aligned} \tag{2.5}
$$

supplemented by

$$
\begin{aligned}
&[e_4, e_5] = -(q - q^{-1}) e_2 e_6 \\
&[e_4, e_6]_{q^{-1}} = 0 \qquad [e_5, e_6]_q = 0.
\end{aligned} \tag{2.6}
$$

Further, we obtain

$$\begin{array}{ll} [e_1, e_{-5}] = 0 & [e_2, e_{-6}] = 0 \\ [e_2, e_{-4}] = e_{-1}q^{h_2} & [e_3, e_{-5}] = e_{-2}q^{h_3} \\ [e_3, e_{-6}] = e_{-4}q^{h_3} & [e_4, e_{-1}] = e_{-2}q^{h_1} \\ [e_4, e_{-3}] = 0 & [e_4, e_{-5}] = (q - q^{-1})q^{-h_2}e_{-3}e_1 \\ [e_5, e_{-2}] = -e_3q^{h_2} & [e_5, e_{-6}] = e_{-1}q^{h_2+h_3} \\ [e_6, e_{-1}] = -e_{-5}q^{h_1} & [e_6, e_{-4}] = e_3q^{h_1+h_2}. \end{array} \tag{2.7}$$

If we add to the relations (2.5)–(2.7), the conjugated ones obtained by means of the involutive antiautomorphism $h_j \to h_j$, $e_{\pm j} \to e_{\mp j}$, $q \to q^{-1}$ we obtain the complete set of relations describing the Cartan–Weyl basis of $U_q(sl(4; \mathbb{C}))$.

The formulae for coproducts are the following (this choice is not unique; see e.g. equation (2.12)):

$$\begin{aligned} &\Delta(e_{\pm j}) = e_{\pm j} \otimes k_j + k_j^{-1} \otimes e_{\pm j} \\ &\Delta(k_j^{\pm 1}) = k_j^{\pm 1} \otimes k_j^{\pm 1} \qquad j = 1, 2, 3 \end{aligned} \tag{2.8}$$

which imply

$$\begin{aligned} \Delta(e_4) &= e_4 \otimes k_4 + k_4^{-1} \otimes e_4 + (1 - q^2)k_1^{-1}e_2 \otimes e_1k_2 \\ \Delta(e_{-4}) &= e_{-4} \otimes k_4 + k_4^{-1} \otimes e_{-4} + (1 - q^{-2})k_2^{-1}e_{-1} \otimes e_{-2}k_1 \\ \Delta(e_5) &= e_5 \otimes k_5 + k_5^{-1} \otimes e_5 + (1 - q^2)k_2^{-1}e_3 \otimes e_2k_3 \\ \Delta(e_{-5}) &= e_{-5} \otimes k_5 + k_5^{-1} \otimes e_{-5} + (1 - q^{-2})k_3^{-1}e_{-2} \otimes e_{-3}k_2 \\ \Delta(e_6) &= e_6 \otimes k_6 + k_6^{-1} \otimes e_6 + (1 - q^2)(k_1^{-1}e_5 \otimes e_1k_5 + k_4^{-1}e_3 \otimes e_4k_3) \\ \Delta(e_{-6}) &= e_{-6} \otimes k_6 + k_6^{-1} \otimes e_{-6} + (1 - q^{-2})(k_5^{-1}e_{-1} \otimes e_{-5}k_1 + k_3^{-1}e_{-4} \otimes e_{-3}k_4) \end{aligned} \tag{2.9}$$

where $k_A = q^{-\frac{1}{2}h_A}$, $A = 1 \ldots 6$.

The formulae for the antipode

$$\begin{array}{lll} S(e_{\pm j}) = -q^{\mp 1}e_{\pm j} & (j = 1, 2, 3) & S(e_{\pm 5}) = q^{\mp 2}\tilde{e}_{\pm 5} \\ S(e_{\pm 4}) = q^{\mp 2}\tilde{e}_{\pm 4} & & S(e_{\pm 6}) = -q^{\mp 3}\tilde{e}_{\pm 6} \end{array} \tag{2.10}$$

introduce the generators of anitpode-extended Cartan–Weyl basis:

$$\begin{array}{ll} \tilde{e}_4 \equiv [e_2, e_1]_q & \tilde{e}_{-4} \equiv [e_{-1}, e_{-2}]_{q^{-1}} \\ \tilde{e}_5 \equiv [e_3, e_2]_q & \tilde{e}_{-5} \equiv [e_{-2}, e_{-3}]_{q^{-1}} \\ \tilde{e}_6 \equiv [\tilde{e}_5, e_1]_q & \tilde{e}_{-6} \equiv [e_{-1}, \tilde{e}_{-5}]_{q^{-1}}. \end{array} \tag{2.11}$$

One can extend the relations for Cartan–Weyl basis by adding the generators (2.11).

It should be mentioned that we introduce the new generators

$$E_{\pm A} = q^{\pm h_A/2}e_{\pm A} \tag{2.12}$$

the coproducts are given by the following non-symmetric form:

$$\begin{aligned}\Delta(E_i) &= E_i \otimes 1 + q^{h_i} \otimes E_i \\ \Delta(E_{-i}) &= E_{-i} \otimes 1 + q^{-h_i} \otimes E_{-i}\end{aligned} \tag{2.13}$$

and the formulae (2.1) and (2.4) can be written for the generators $E_{\pm A}$ if $[x]_q \equiv (1-q^{2x})/(1-q^2)$. It appears, for example, that the coproducts (2.13) were used in [11] in the derivation of the universal R-matrix formula. One can also think of coproduct (2.13) as obtained from (2.8) by twisting [12].

3. Standard real forms of $U_q(sl(4;\mathbb{C}))$

Twietmeyer [8] obtained a complete list of standard real forms of $U_q(\mathcal{G})$, where $\mathcal{G}$ is any simple complex Lie algebra. By a standard real form we mean here a morphism Φ: $U_q(\mathcal{G}) \to U_q(\mathcal{G})$ with the following properties ($X, Y \in \mathcal{G}$, $\mu, \nu \in \mathbb{C}$):†

$$\Phi^2 = \mathbf{1} \tag{3.1a}$$

$$\Phi(XY) = \Phi(Y)\Phi(X) \tag{3.1b}$$

$$\Phi(\mu X + \nu Y) = \mu^*\Phi(X) + \nu^*\Phi(Y) \tag{3.1c}$$

$$\Delta \circ \Phi = (\Phi \otimes \Phi) \circ \Delta \tag{3.1d}$$

which imply

$$\Phi \circ S \circ \Phi \circ S = \mathbf{1} \tag{3.1e}$$

$$\epsilon(\Phi(X)) = (\epsilon(X))^*. \tag{3.1f}$$

Twietmeyer's list was given for the specific choice of the coproduct in $U_q(\mathcal{G})$—that given for $U_q(sl(4;c))$ by (2.13). It can however be shown that for other coproducts one obtains analogous classification and our results below are given for the coproduct (2.8). As $q \to 1$ they in fact become the same.

In applications one must be able to recognize which real form of the complex Lie algebra $\mathcal{G}$ does correspond to Φ. For this purpose we would like to remind some basic facts from the theory of real Lie algebras. A fundamental theorem (see e.g. [13]) tells us that real forms of a complex simple Lie algebra $\mathcal{G}$ are described by means of involutive automorphisms of $\mathcal{G}$, i.e. mnorphisms Ψ satisfying:

$$\begin{aligned}&\Psi^2 = 1 \\ &\Psi([X,Y]) = [\Psi(X), \Psi(Y)] \\ &\Psi(\mu X + \nu Y) = \mu\Psi(X) + \nu\Psi(Y).\end{aligned} \tag{3.3}$$

The simplest method to investigate which two transformations with the above properties describe isomorphic real Lie algebras is based on the analysis of the action of Ψ on the (unique) compact real form of $\mathcal{G}$. When diagonalized Ψ can only have eigenvalues ± 1. If

† According to [5] the standard real forms are described by $*$-involution. The relations (3.1b) and (3.1e) are written already as relations (1.2) (first one) and (1.3) second one).

'negative' eigenvectors are multiplied by i we obtain a real Lie algebra G_Ψ with generators satisfying the reality condition $\Psi(X) = X$. Two Ψ's with different spectrum define non-isomorphic real Lie algebras (see e.g. [13]).

In the case of $sl(4;\mathbb{C})$ spectrum of Ψ characterizes real forms completely. If we look at five real forms of that Lie algebra we quickly recognize that corresponding Φ's have the following number of minus signs in their spectrum (in the action on $su(4)$):

$$\begin{aligned} &su(4) \sim 0 \qquad su(3,1) \sim 6 \\ &sl(4;\mathbb{R}) \sim 9 \qquad su^*(4) \sim 5 \\ &su(2,2) \sim 8. \end{aligned} \tag{3.4}$$

As $q \to 1$ every antiautomorphism Φ given in (3.1) defines a real form of $sl(4;\mathbb{C})$—it consists from elements satisfying the condition $\Phi(X) = -X$. It is easy to find a corresponding automorphism Ψ which gives rise to the same set of real generators. If we look now at its spectrum we immediately recognize what real form of $sl(4;\mathbb{C})$ is described by Φ. Twietmeyer obtained 12 non-equivalent real forms in that case. As $q \to 1$ they correspond to all five real forms of $sl(4;\mathbb{C})$. Both numbers disagree as we obtain four non-isomorphic quantum deformations of $U_q(su(2;2)) \simeq U_q(o(4,2))$, two deformations of $U_q(sl(4;\mathbb{R}) \simeq U_q(o(3,3))$ and four deformations of $su(3,1)$; only deformations of $U_q(su(4)) \simeq U_q(o(6))$ and $U_q(su^*(4)) \simeq U_q(o(5,1))$ have unique (up to equivalence) real forms. Here we reproduce a complete list of real forms of $U_q(sl(4;\mathbb{C})$ (for a coproduct given in equation (2.8); $j = 1,2,3$):

$$\begin{array}{llll} |q| = 1 & \Phi_1(h_j) = -h_{4-j} & \Phi_1(e_{\pm j}) = e_{\pm(4-j)} & U_q(o(4,2)) \\ |q| = 1 & \Phi_2(h_j) = -h_j & \Phi_2(e_{\pm j}) = e_{\pm j} & U_q(o(3,3)) \\ q \in \mathbb{R} & \Phi_3(h_j) = h_{4-j} & \Phi_3(e_{\pm j}) = e_{\mp(4-j)} & U_q(o(5,1)) \\ q \in \mathbb{R} & \Phi_4(h_j) = h_{4-j} & \Phi_4(e_{\pm j}) = (-)^{\delta_{j,2}} e_{\mp(4-j)} & U_q(o(3,3)) \\ q \in \mathbb{R} & \Phi_5(h_j) = h_j & \Phi_5(e_{\pm j}) = \epsilon_j e_{\mp j} & (\epsilon_j = \pm 1). \end{array} \tag{3.5}$$

The real forms described by Cartan involutions Φ_5 are given in the following table:

Table 1. Standard Cartan $+-$ involutions for $U_q(sl(4;c))$.

ϵ_1	ϵ_2	ϵ_3	Real form
1	1	1	$U_q(o(6)) \simeq U_q(su(4))$
−1	1	1	$U_q(su(3,1))$
1	−1	1	$U_q(o(4,2)) \simeq U_q(su(2,2))$
−1	−1	1	$U_q(su(3,1))$
1	1	−1	$U_q(su(3,1))$
−1	1	−1	$U_q(o(4,2)) \simeq U_q(su(2,2))$
1	−1	−1	$U_q(su(3,1))$
−1	−1	−1	$U_q(o(4,2)) \simeq U_q(su(2,2))$

The morphisms Φ_k ($k = 1\ldots5$) are completely defined by its action on simple root generators of $U_q(sl(4;c))$. In particular we can deduce how does Φ act on other generators of the Cartan–Weyl basis. Let us take for example $e_4 = [e_1, e_2]_q$. In five cases listed in equation (3.5) we obtain:

$$\begin{aligned} &\Phi_1(e_4) = -q^{-1}\tilde{e}_5 \qquad \Phi_2(e_4) = -q^{-1}e_4 \\ &\Phi_3(e_4) = -qe_{-5} \qquad \Phi_4(e_4) = qe_{-5} \\ &\Phi_5(e_4) = -q\epsilon_1\epsilon_2\tilde{e}_{-4}. \end{aligned} \tag{3.6}$$

Therefore only in the case of real forms defined by means of morphisms Φ_2, Φ_3, Φ_4 does its action not lead us outside the Cartan–Weyl basis. On the other hand Φ_1 and Φ_5 inevitably leads us to the antipode-extended Cartan–Weyl basis defined in (2.11).

We see that deformation of the conformal algebra $su(2;2)$ can only be achieved by means of morphisms Φ_1 and Φ_5, with $\epsilon_1 = \epsilon_3 = -1$ or $\epsilon_1 = \epsilon_3 = 1$ and $\epsilon_2 = -1$.

Let us write the action of Φ_1, Φ_5 on the Cartan–Weyl basis of $U_q(sl(4;c))$, providing real conformal algebras $U_q(o(4,2))$:

$$\begin{aligned} &\Phi_1(e_1) = e_3 \qquad \Phi_1(e_2) = e_2 \qquad \Phi_1(e_4) = -q^{-1}\tilde{e}_5 \\ &\Phi_1(e_5) = -q^{-1}\tilde{e}_4 \qquad \Phi_1(e_6) = q^{-2}\tilde{e}_6 \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\Phi_5(e_1) = \epsilon e_{-1} \qquad \Phi_5(e_2) = \epsilon_2 e_{-2} \qquad \Phi_5(e_3) = \epsilon e_{-3} \\ &\Phi_5(e_4) = -\epsilon\epsilon_2 q\tilde{e}_{-4} \qquad \Phi_5(e_5) = -\epsilon\epsilon_2 q\tilde{e}_{-5} \qquad \Phi_5(e_6) = -\epsilon_2 q\tilde{e}_{-6} \end{aligned} \tag{3.8}$$

where $\epsilon = \epsilon_1 = \epsilon_3$, $(\epsilon, \epsilon_2) = (1, -1), (-1, 1)$ or $(-1, -1)$. We obtain in such a way four real $D = 4$ conformal quantum algebras: one $U_q^{\Phi_1}(o(4,2))$ and three $U_q^{\Phi_5}(o(4,2))$. One can show that:

(i) If we use the real form (3.7), the generators $M_{\mu\nu} = (h_1, h_3, e_{\pm 1}, e_{\pm 3})$ generate the real Hopf subalgebra $U_q(o(3,1))$, i.e. we have

$$U_q(o(3,1)) \subset U_q^{\Phi_1}(o(4,2)). \tag{3.9}$$

Unfortunately from the remaining generators ($e_{\pm 2}, e_{\pm a}, \tilde{e}_{\pm a}$; $a = 4, 5, 6$) one can not form the real four-momenta generators which form, together with six generators $M_{\mu\nu}$, a closed subalgebra of $U_q^{\Phi_1}(o(4,2))$.

(ii) For the real form (3.8) and $\epsilon = 1$, $\epsilon_2 = -1$ the generators $M_{\mu\nu}$ describe real q-deformed $o(4)$ Hopf algebra, i.e. we obtain

$$U_q(su(2)) \oplus U_q(su(2)) \equiv U_q(o(4)) \subset U_q^{\Phi_5}(o(4,2)). \tag{3.10a}$$

(iii) For two real forms (3.8) with $\epsilon = -1$ ($\epsilon_2 = \pm 1$) the generators $M_{\mu\nu}$ describe the real q-deformed $o(2,2)$ algebra, i.e. we get

$$U_q(su(1,1)) \oplus U_q(su(1,1)) \equiv U_q(o(2,2)) \subset U_q^{\Phi_5}(o(4,2)). \tag{3.10b}$$

We see that for physical applications the best choice is described by the real form (3.7), i.e. we obtain real quantum conformal algebra with the quantum Lorentz algebra as its real Hopf subalgebra. If we wish to obtain the sequence (1.5) of real Hopf algebras, one has to consider the $\oplus$-involutions firstly proposed in [2].

4. All standard and non-standard real forms of $U_q(sl(4;c))$

Let us introduce (besides (1.1) and (1.2)) the remaining two involutive automorphisms which describe the automorphisms in the multiplication sector

$$
\begin{aligned}
&(a\cdot b)^* = a^*\cdot b^* &\qquad (\Delta(a))^* = \Delta'(a^*)\\
&(a\cdot b)^{\circledast} = a^{\circledast}\cdot b^{\circledast} &\qquad (\Delta(a))^{\circledast} = \Delta(a^{\circledast}).
\end{aligned}
\tag{4.1}
$$

We would like to introduce the basic involutive automorphisms of $U_q(sl(4;c))$ from which we will be able to construct all the morphisms describing standard and non-standard real forms of $U_q(sl(4;c))$. They are given by the following four mappings of $U_q(sl(4;c))$ Hopf algebra into itself.

(i) Q-automorphism which is $*$-involution changing $q\to q^{-1}$†

$$Q(e_{\pm i}) = e_{\pm i} \qquad Q(h_i) = h_i \qquad i=1,2,3 \tag{4.2a}$$

$$Q(e_{\pm 4}) = -q^{\mp 1}\tilde{e}_{\pm 4} \qquad Q(e_{\pm 5}) = -q^{\mp 1}\tilde{e}_{\pm 5} \tag{4.2b}$$

$$Q(e_{\pm 6}) = q^{\mp 2}\tilde{e}_{\pm 6} \qquad Qq = q^{-1}. \tag{4.2c}$$

(ii) Ω-automorphism which is $\circledast$-involution, exchanging the first and third root in the Dynkin diagram of $sl(4;\mathbb{C})$ is not changing q‡

$$
\begin{aligned}
&\Omega(e_{\pm 1}) = e_{\pm 3} \qquad \Omega(h_1) = h_3\\
&\Omega(e_{\pm 2}) = e_{\pm 2} \qquad \Omega(h_2) = h_2
\end{aligned}
\tag{4.3}
$$

$$\Omega e_{\pm 4} = \tilde{e}_{\pm 5} \qquad \Omega e_{\pm 5} = \tilde{e}_{\pm 4} \tag{4.3b}$$

$$\Omega e_{\pm 6} = \tilde{e}_{\pm 6} \qquad \Omega q = q. \tag{4.3c}$$

(iii) The transposition mapping T, which is $\oplus$-involution not changing q:

$$T(e_{\pm i}) = e_{\pm i} \qquad T(h_i) = -h_i \tag{4.4a}$$

$$T(e_{\pm 4}) = \tilde{e}_{\pm 4} \qquad Te_{\pm 5} = \tilde{e}_{\pm 5} \tag{4.4b}$$

$$T(e_{\pm 6}) = \tilde{e}_{\pm 6} \qquad Tq = q. \tag{4.4c}$$

The transposition mapping T satisfying $T^2=1$ differs only by numerical factor with the antipode S (which does not satisfy the $S^2=1$ condition).

(iv) Standard Cartan $+$-involutions ($\Phi^5 \equiv C(\epsilon_1,\epsilon_2,\epsilon_3)$; (see (3.5)). One gets (see also table 1):

$$C(e_{\pm i}) = \epsilon_i e_{\mp i} \qquad Ch_i = h_i \tag{4.5a}$$

$$Ce_{\pm 4} = -\epsilon_1\epsilon_2 q^{\pm 1}\tilde{e}_{\mp 4} \qquad Ce_{\pm 5} = -\epsilon_2\epsilon_3 q^{\pm 1}\tilde{e}_{\mp 5} \tag{4.5b}$$

$$Ce_{\pm 6} = \epsilon_1\epsilon_2\epsilon_3 q^{\pm 2}\tilde{e}_{\mp 6} \qquad Cq = q. \tag{4.5c}$$

† This involution denoted by σ we found in [14].

‡ This involution was suggested by V N Tolstoy.

The complete set of the involutive automorphism of $U_q(sl(4;c))$ Hopf algebra is given by the following seven types of involutions

$$Q,\ \Omega,\ T,\ Q\Omega,\ QT,\ \Omega T,\ Q\Omega T \tag{4.6}$$

possibly multiplied by the involution $C(\epsilon_1, \epsilon_2, \epsilon_3)$. Observe however that as we wish to obtain involutions (morphisms which square equals identity) we must check whether basic involutions commute among themselves. It is not always true in a case of C and Ω. Whenever we combine these two morphisms we should always assume that $\epsilon_1 = \epsilon_3$ which implies $[C, \Omega] = 0$. Furthermore it turns out [8] that if combined with Ω only $C(1,1,1)$ and $C(1,-1,1)$ give rise to non-isomorphic real forms.

Using the multiplication rules for the four types of involutions (note that identity is an operation of type $\circledast$) we see that:

Table 2. Multiplication table for different types of involutions.

	$+$	$\oplus$	$*$	$\circledast$
$+$	1	$*$	$\oplus$	$+$
$\oplus$	$*$	?	$+$	$\oplus$
$*$	$\oplus$	$+$	1	$*$
$\circledast$	$+$	$\oplus$	$*$	1

(i) 12 standard $+$-involutions are given by $Q \circ T$, $Q \circ \Omega \circ T$, $C(\epsilon_1, \epsilon_2, \epsilon_3)$ and $\Omega \circ C(\epsilon_1, \epsilon_2, \epsilon_3)$. It appears that out of eight involutions $\Omega \circ C$ only two (e.g. $\Omega \circ C(1,1,1)$ and $\Omega \circ C(1,-1,1)$) are non-quivalent. Comparing with section 3 one can check that

$$\begin{array}{lll} \Phi_1 = Q \circ \Omega \circ T & \Phi_2 = Q \circ T & |q| = 1 \\ \Phi_3 = \Omega \circ C(1,1,1) & \Phi_4 = \Omega \circ C(1,-1,1) & q \text{ real} \\ \Phi_5 = C(\epsilon_1, \epsilon_2 \epsilon_3) & & q \text{ real.} \end{array} \tag{4.7}$$

(ii) 12 non-standard $\oplus$-involutions are obtained by multiplying the standard involutions (4.7) by Q treated as a complex-linear mapping ($Q(\alpha A) = \alpha Q(A)$ for α complex). Because Q describes the mapping $q \to q^{-1}$, the conditions $|q| = 1$ (q real) in (4.7) are replaced by q real ($|q| = 1$). One obtains

$$\begin{array}{lll} \tilde\Phi_1 = \Omega \circ T & \tilde\Phi_2 = T & q \text{ real} \\ \tilde\Phi_3 = Q \circ \Omega \circ C(1,1,1) & \tilde\Phi_4 = Q \circ \Omega \circ \omega \circ C(1,-1,1) & |q| = 1 \\ \tilde\Phi_5 = Q \circ C(\epsilon_1, \epsilon_2, \epsilon_3) & & |q| = 1. \end{array} \tag{4.8}$$

Two $\oplus$-involutions considered in [2] providing $U_q(o(4,2))$ as real $\oplus$-Hopf algebra are given by $\tilde\Phi_1$ and $\tilde\Phi_4$.

(iii) 12 non-standard $*$-involutions are obtained by multiplying the involutions (4.7) by T, treated as a complex-linear mapping. One obtains

$$\begin{array}{lll} \Psi_1 = Q \circ \Omega & \Psi_2 = Q & |q| = 1 \\ \Psi_3 = T \circ \Omega \circ C(1,1,1) & \Psi_4 = T \circ \Omega \circ C(1,-1,1) & q \text{ real} \\ \Psi_5 = T \circ C(\epsilon_1, \epsilon_2, \epsilon_3) & & q \text{ real.} \end{array} \tag{4.9}$$

(iv) 12 non-standard $\circledast$-involutions (the twelfth is an identity mapping) is obtained by multiplying (4.7) by $Q \circ T$, treated as complex-linear mapping

$$\begin{array}{lll} \tilde{\Psi}_1 = \Omega \circ T & \tilde{\Psi}_2 = 1 & q \text{ real} \\ \tilde{\Psi}_3 = Q \circ T \circ \Omega \circ C(1,1,1) & \tilde{\Psi}_4 = Q \circ T \circ \Omega \circ C(1,-1,1) & |q| = 1 \\ \tilde{\Psi}_5 = Q \circ T \circ C(\epsilon_1, \epsilon_2, \epsilon_3) & & |q| = 1. \end{array} \tag{4.10}$$

In such a way all the involutions (4.7)–(4.10) are antilinear complex mappings (see (3.1*c*)).

5. Real forms of $U_q(sl(4;\mathbb{C}))$ and the universal R-matrix

Using the uniqueness theorem of Khoroshkhin and Tolstoy [15] one can immediately see what is a result of the action of four mappings introduced in the section 2 on the universal R-matrix. Let us remind that according to the above mentioned theorem an element $R \in U_q(sl(4;\mathbb{C})) \otimes U_q(sl(4;\mathbb{C}))$ satisfying two conditions:

(i) $\Delta'(a) = R \circ \Delta(a) \circ R^{-1}$,

(ii) $R \in T_q(b_+ \otimes b_-)$ (i.e. R belongs to the so-called Taylor extension of $U_q(b_+) \otimes U_q(b_-)$, see [15])

is unique up to multiplicative constant. For a certain value of that constant R satisfies also

$$(\Delta \otimes 1) \circ R = R_{13} \circ R_{23} \qquad (1 \otimes \Delta) \circ R = R_{13} \circ R_{12}. \tag{5.1}$$

The explicit construction of the universal R-matrix satisfying conditions (i) and (ii) was given in [9, 11, 16]. Here we would like to investigate the question what is $Q(R)$, $\Omega(R)$, $T(R)$, $C(R)$ with Q, Ω, T, C introduced in section 2. We derive first:

$$\begin{aligned} \Delta'(Q(a)) &= Q(R^{-1}) \circ \Delta(Q(a)) \circ Q(R) \\ \Delta'(\Omega(a)) &= \Omega(R) \circ \Delta(\Omega(a)) \circ \Omega(R^{-1}) \\ \Delta'(T(a)) &= T(R) \circ \Delta(T(a)) \circ T(R^{-1}) \\ \Delta'(C(a)) &= C(R^{-1}) \circ \Delta(C(a)) \circ C(R). \end{aligned} \tag{5.2}$$

We observe that for $X = Q, \Omega, T, C$

$$\{X(a),\ a \in U_q(sl(4;\mathbb{C}))\} = U_q(sl(4;\mathbb{C})). \tag{5.3}$$

It is also clear that for $Z = Q, \Omega, T$

$$Z(b_\pm) \subset b_\pm \tag{5.4a}$$

and

$$C(b_\pm) \subset b_\mp. \tag{5.4b}$$

From that we immediately obtain that

$$\begin{array}{ll} Q(R) = R^{-1} & T(R) = R \\ \Omega(R) = R & C(R) = \tau \circ R. \end{array} \tag{5.5}$$

In particular for five real forms listed by Twietmeyer we obtain:

$$\begin{aligned} &\Phi_1(R) = R^{-1} \qquad \Phi_2(R) = R^{-1} \qquad \Phi_3(R) = \tau \circ R \\ &\Phi_4(R) = \tau \circ R \qquad \Phi_5(R) = \tau \circ R. \end{aligned} \tag{5.6}$$

Let us remind that for the first two morphisms $|q| = 1$, while for the last three $q \in \mathbb{R}$.

The relations (5.6) can also be checked explicitly if we observe that the formula for the universal $U_q(sl(4;c))$ R-matrix has a form (see e.g. [11])

$$\begin{aligned} R &= R_{E_1} R_{E_4} R_{E_6} R_{E_2} R_{E_5} R_{E_3} \cdot K \\ &= R_{E_3} R_{\tilde{E}_5} R_{E_2} R_{\tilde{E}_6} R_{\tilde{E}_4} R_{E_1} \cdot K \end{aligned} \tag{5.7}$$

where the root generators $E_{\pm A}$ are given by (2.12) and

$$R_{E_k} = \exp_{q^{-2}}[(q - q^{-1}) E_k \otimes E_{-k}] \tag{5.8a}$$

$$K = q^{d_{ij}} h_i \otimes h_j \qquad (i, j = 1, 2, 3) \tag{5.8b}$$

where

$$\exp_q(x) \equiv \sum_{n \geqslant 0} \frac{x^n}{(n)_q!} \tag{5.9a}$$

$$(n)_q! \equiv (1)_q (2)_q \ldots (n)_q \tag{5.9b}$$

$$(k)_q \equiv \frac{1 - q^k}{1 - q} \tag{5.9c}$$

and d_{ij} is the inverse matrix for symmetrical Cartan matrix given in (2.1).

The relation (5.7) has been written in two equivalent forms, corresponding to the clockwise and anti-clockwise normal order of root generators [10, 11].

6. Final remarks

The aim of this paper is to show that the standard reality conditions impose severe restrictions on the choice of quantum deformations of real $D = 4$ conformal algebra. In particular it does not exist real $D = 4$ quantum Weyl algebra with standard reality conditions obtained as Hopf subalgebra of $U_q(sl(4;c))$. In such a way the proposal presented in [2] can not be improved.

It should be mentioned here that another quantum deformation of $D = 4$ quantum Weyl algebra, with standard reality condition has been recently derived from the different realizations on q-deformed $D = 4$ Minkowski space [17]. Unfortunately we were not able to find the formulae relating these two deformations—one obtained in purely algebraic way, and the second abstracted from a concrete realization in the framework of non-commutative differential geometry.

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